Section 16.3:
The Fundamental Theorem for Line Integrals

What We'll Learn In Section 16.3

1. When is a Vector Field on $\mathbb{R}^2$ Conservative?
2. Finding the Potential Function for a Conservative Vector Field
3. The Fundamental Theorem for Line Integrals
4. Independence of Path
5. Conservation of Energy in Physics

1. When is a Vector Field on $\mathbb{R}^2$ Conservative?

Recall:

- A vector field $\vec{F}$ is called a conservative vector field if it is the gradient of some scalar function.
- That is, $\vec{F}$ is conservative if there is a scalar function f such that $\nabla f = \vec{F}$.
- In this situation f is called a potential function for $\vec{F}$.

5 Theorem

If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first- order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Notes:

- This result is only for vector fields on $\mathbb{R}^2$
- The other way around is false
- This result is only really useful in showing that a vector field is NOT conservative

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<u>Proof</u>:

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Ex 1: Show that $\vec{F} =< x^2y,\ x + xy >$ is NOT a conservative vector field.

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$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Ex 2:

a) Show that $\vec{F} =< x^2y,\ \frac{1}{3}x^3 + \sin y >$ satisfies the above condition.

b) Does part (a) on its own mean that $\vec{F}$ is conservative?

1. When is a Vector Field on $\mathbb{R}^2$ Conservative?

Def:

1) A subset D of $\mathbb{R}^2$ is open if given any point in D you can draw a disk around the point that is entirely contained in D.
   That is, D does not contain any of its boundary points.
2) A subset D of $\mathbb{R}^2$ is connected if any 2 points in D can be joined by a path entirely contained in D.
3) A subset D of $\mathbb{R}^2$ is simply-connected if D has no holes in it.

6 Theorem

Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

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<u>Proof</u>:

From Green's Theorem (next section) and Theorem 4 (end of this section)

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$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

Ex 2 (again):

a) Show that $\vec{F} =< x^2y, \frac{1}{3}x^3 + \sin y >$ satisfies the above condition.

b) Does part (a) on its own mean that $\vec{F}$ is conservative?

2. Finding the Potential Function for a Conservative Vector Field

<u>Ex 3</u>: Consider the vector field $\vec{F} =< x - y, x - 2 >$.

a) Determine if this vector field is conservative.

b) If $\vec{F}$ is conservative, find a function *f* such that $\nabla f = \vec{F}$. That is, find the vector field's potential function.

2. Finding the Potential Function for a Conservative Vector Field

Ex 4: Consider the vector field $\vec{F} =< 3 + 2xy,\ x^2 - 3y^2 >$.

a) Determine if this vector field is conservative.

b) If $\vec{F}$ is conservative, find a function *f* such that $\nabla f = \vec{F}$. That is, find the vector field's potential function.

2. Finding the Potential Function for a Conservative Vector Field

Ex 5: The vector field $\vec{F} =< y^2,\ 2xy + e^{3z},\ 3ye^z >$ is conservative. Find a function *f* such that $\nabla f = \vec{F}$. That is, find the vector field's potential function.

3. The Fundamental Theorem for Line Integrals

2 Theorem

Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

<u>Proof ???:</u>

3. The Fundamental Theorem for Line Integrals

Ex 4 (again): Consider the vector field $\vec{F} =< 3 + 2xy,\ x^2 - 3y^2 >$.

a) Determine if this vector field is conservative.

b) If $\vec{F}$ is conservative, find a function *f* such that $\nabla f = \vec{F}$. That is, find the vector field's potential function.

c) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where *C* is the curve given by

$$\vec{r}(t) =< e^t \sin t\,, e^t \cos t > \qquad 0 \le t \le \pi$$

3. The Fundamental Theorem for Line Integrals

Ex 5 (again):

a) The vector field $\vec{F} =< y^2,\ 2xy + e^{3z},\ 3ye^z >$ is conservative. Find a function f such that $\nabla f = \vec{F}$. That is, find the vector field's potential function.

b) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is any space curve that starts at (0,1,1) and ends at (2, 2, 0).

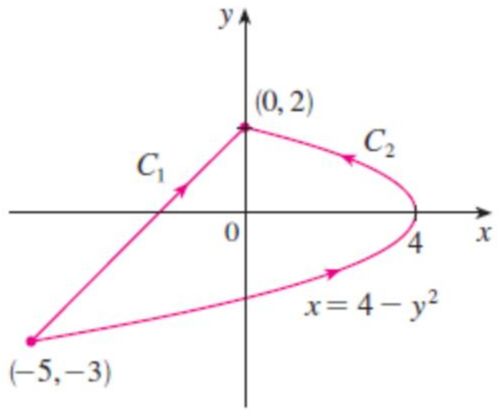
3. The Fundamental Theorem for Line Integrals

Ex 6: Find the work done by the gravitational field $\vec{F}(\vec{x}) = -\frac{mMG}{|\vec{x}|^3}\vec{x}$ in moving a particle with mass *m* from point (3,4,12) to the point (2,2,0) along a piecewise smooth curve *C*.

(Recall: $\vec{F}$ is conservative and its potential function is $f(x, y, z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$)

4. Independence of Path

- The line integral $\int_C \vec{F} \cdot d\vec{r}$ is <u>independent of path</u> if given any 2 curves C_1 and C_2 with the same starting and ending points, $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$



$$\vec{F} = \langle y^2, x \rangle$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = -\frac{5}{6} \qquad \int_{C_2} \vec{F} \cdot d\vec{r} = \frac{245}{6}$$

- The Fundamental Theorem for Line Integrals tells us that conservative vector fields are independent of path.

$$\int_C \nabla f \cdot d\vec{r} = f(\text{ending point}) - f(\text{starting point})$$

4. Independence of Path

- A curve C is a closed curve if its starting and ending points are the same.

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3 Theorem

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

4. Independence of Path

3 Theorem

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Proof:

4. Independence of Path

4 Theorem

Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Proof:

5. Conservation of Energy in Physics

Suppose an object of mass m is under the influence of a conservative force field $\vec{F}(x, y, z) = \nabla f(x, y, z)$ as it travels in space from point A to point B. …